

QUASI-ALTERNATING LINKS AND  $Q$ -POLYNOMIALS

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ABSTRACT. Qazaqzeh and Chbili showed that for any quasi-alternating link, the degree of  $Q$ -polynomial is less than its determinant. We give a refinement of their evaluation.

## 1. INTRODUCTION

The notion of quasi-alternating links was introduced by Ozsváth and Szabó [14], and it is recognized as one of important classes of links in knot theory. For example, see [4, 6, 7, 12, 16, 17, 20]. We recall the definition of quasi-alternating links.

The set of  $\mathcal{Q}$  of *quasi-alternating links* is the smallest set of links which satisfies the following properties.

- (1) The unknot is in  $\mathcal{Q}$ .
- (2) Let  $L$  be a link whose diagram  $D$  has a crossing  $c$  such that
  - (a) both resolutions  $L_\infty$  and  $L_0$ , obtained from  $D$  by smoothing the crossing  $c$  as in Figure 1, lie in  $\mathcal{Q}$ ; and
  - (b)  $\det L = \det L_\infty + \det L_0$ .

Then  $L$  lies in  $\mathcal{Q}$ .

Such a crossing  $c$  is called a *quasi-alternating crossing*.

Alternating knots and non-split alternating links are quasi-alternating [14]. However, it is not an easy task to determine whether a given knot or link is quasi-alternating or not, in general. For example, Greene [6] showed that double branched covers do not bound negative definite 4-manifolds without homological torsion in order to prove that the targets are not quasi-alternating. Also, knot Floer homology and Khovanov homology are known to be an obstruction to a link being quasi-alternating [12].

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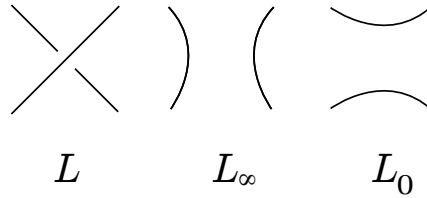


FIGURE 1. Two resolutions of  $L$

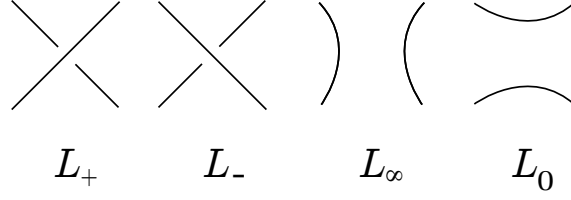


FIGURE 2. Resolutions

On the other hand, Qazaqzeh and Chbili [16] found a very simple constraint on the highest degree of  $Q$ -polynomial for quasi-alternating links.

For unoriented links,  $Q$ -polynomials were introduced by [2] and [8]. Let  $L$  be an unoriented link. Its  $Q$ -polynomial  $Q_L$  is a Laurent polynomial in  $\mathbb{Z}[x, x^{-1}]$ , defined as follows.

- (1) For the unknot  $U$ ,  $Q_U = 1$ .
- (2)  $Q_{L_+} + Q_{L_-} = x(Q_{L_\infty} + Q_{L_0})$ , where  $L_+, L_-, L_\infty, L_0$  are four links which are identical except in a small region where they look like as in Figure 2.

**Theorem 1.1** (Theorem 1.2 of [16]). *For any quasi-alternating link  $L$ ,*

$$\deg Q_L \leq \det L - 1.$$

For example, the knot  $8_{19}$ , which is the torus knot of type  $(3, 4)$ , has determinant 3, but the degree of its  $Q$ -polynomial is 6. Thus  $8_{19}$  is not quasi-alternating.

In general, for any link but the unknot, the degree of its  $Q$ -polynomial is less than the crossing number ([2]). And, it is a classical fact that the crossing number is less than or equal to the determinant for any non-split alternating link ([1, 5]). Thus Theorem 1.1 can be seen as a natural generalization of the same evaluation for non-split alternating links.

The purpose of this short paper is to give a slight improvement of the evaluation by Qazaqzeh and Chbili [16].

**Theorem 1.2.** *Let  $L$  be a quasi-alternating link. If  $L$  is not a  $(2, n)$ -torus link, then*

$$\deg Q_L \leq \det L - 2.$$

Of course, the  $(2, n)$ -torus link  $L$  is alternating, so quasi-alternating, unless  $n = 0$ . It has determinant  $|n|$ , but it is easy to show that  $\deg Q_L = |n| - 1$ . Thus the conclusion of Theorem 1.2 does not hold. Also, the figure-eight knot has determinant 5, and its  $Q$ -polynomial is  $2x^3 + 4x^2 - 2x - 3$ . Since the figure-eight knot is quasi-alternating, the evaluation of Theorem 1.2 is sharp.

In the proof of Theorem 1.2, the Dehn surgery characterization of the unknot by [11, 15] plays a key role.

## 2. PROOF OF THEOREM 1.2

The next lemma is a key step of the proof of Theorem 1.1.

**Lemma 2.1** (Lemma 2.2 of [16]). *Let  $L$  be a link. Then*

$$\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1,$$

*where  $L_0$  and  $L_\infty$  are the resolutions of  $L$  at any crossing.*

**Lemma 2.2.** *Let  $L$  be a quasi-alternating link. If  $\det L = 1, 2$  or  $3$ , respectively, then  $L$  is the unknot, the Hopf link or a trefoil, respectively.*

*Proof.* This is found in the proof of Proposition 3.2 in [6].  $\square$

**Lemma 2.3.** *Let  $L$  be a quasi-alternating link. If  $\det L = 4$ , then  $L$  is the  $(2, \pm 4)$ -torus link or  $\deg Q_L \leq 2$ .*

*Proof.* Let  $c$  be a quasi-alternating crossing of  $L$ . Let  $L_0$  and  $L_\infty$  be two resolutions of  $L$  at the crossing  $c$ . Then both of  $L_0$  and  $L_\infty$  are quasi-alternating. Since  $\det L = \det L_0 + \det L_\infty$ ,  $\{\det L_0, \det L_\infty\} = \{3, 1\}$  or  $\det L_0 = \det L_\infty = 2$ .

First, we may assume that  $\det L_0 = 3$  and  $\det L_\infty = 1$ . By Lemma 2.2,  $L_0$  is a trefoil and  $L_\infty$  is the unknot. Let  $\gamma$  be an unknotted arc connecting the strands at the resolution of  $L_\infty$ , and let  $K$  be the lift of  $\gamma$  in the double branched cover  $\Sigma(L_\infty) = S^3$ . Then  $\Sigma(L_0) = \pm L(3, 1)$  is obtained by an integral Dehn surgery on  $K$ . By [11, Theorem 1.1 and Corollary 8.4] (or [19, Theorem 9]) and [13],  $K$  is the unknot. Hence  $\Sigma(L)$  is also obtained by an integral Dehn surgery on the unknot  $K$ , so  $\Sigma(L) = \pm L(4, 1)$ . This implies that  $L$  is the  $(2, \pm 4)$ -torus link by [9].

Next, assume that  $\det L_0 = \det L_\infty = 2$ . By Lemma 2.2 again, both  $L_0$  and  $L_\infty$  are Hopf links. Note that  $Q_{L_0} = Q_{L_\infty} = 2x + 1 - 2x^{-1}$ . By Lemma 2.1,  $\deg Q_L \leq 2$ .  $\square$

It seems to be open that a quasi-alternating link with determinant 4 should be either the  $(2, \pm 4)$ -torus link or the connected sum of two Hopf links.

*Proof of Theorem 1.2.* The argument is done by induction on determinant of  $L$ . First, we note that  $\det L \geq 4$  by Lemma 2.2, under our assumption.

Suppose that  $\det L = 4$ . By Lemma 2.3 and our assumption,  $\deg Q_L \leq 2$ .

Now, suppose that the conclusion is true for any quasi-alternating link with determinant less than or equal to  $m$  ( $\geq 4$ ), which is not a  $(2, n)$ -torus link. Let  $L$  be a quasi-alternating link with determinant  $m + 1$ . Choose a quasi-alternating crossing  $c$ , and let  $L_0$  and  $L_\infty$  be the resolutions at  $c$ . Then both of the resolutions are quasi-alternating, and the equation  $\det L = \det L_0 + \det L_\infty$  holds. Thus  $L_0$  and  $L_\infty$  have determinant less than or equal to  $m$ .

We split the argument into 3 cases.

(1) Neither  $L_0$  nor  $L_\infty$  is a  $(2, n)$ -torus link.

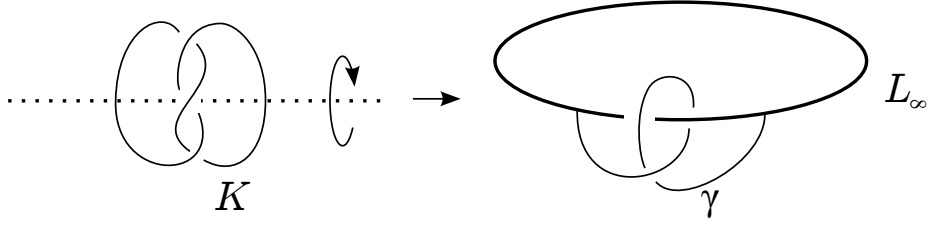
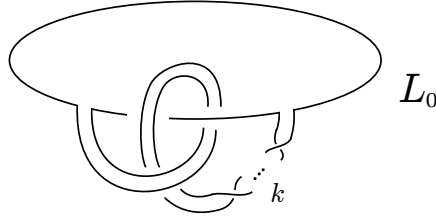
By inductive hypothesis, we have  $\deg Q_{L_0} \leq \det L_0 - 2$  and  $\deg Q_{L_\infty} \leq \det L_\infty - 2$ . Thus,

$$\begin{aligned} \deg Q_L &\leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1 \\ &= \deg Q_{L_*} + 1 \\ &\leq (\det L_* - 2) + 1 \\ &\leq \det L - 2, \end{aligned}$$

where  $*$   $\in \{0, \infty\}$  is chosen appropriately. The last inequality follows from the equation  $\det L = \det L_0 + \det L_\infty$ .

(2) Only one of  $L_0$  and  $L_\infty$  is a  $(2, n)$ -torus link.

We may assume that  $L_0$  is the  $(2, p)$ -torus link. Then  $\det L_0 = |p|$  and  $\deg Q_{L_0} = |p| - 1$ . For  $L_\infty$ , we have  $\deg Q_{L_\infty} \leq \det L_\infty - 2$  by inductive hypothesis. If  $\deg Q_{L_0} \leq \deg Q_{L_\infty}$ , then  $\deg Q_L \leq (\det L_\infty - 2) + 1 \leq \det L - 2$  as in (1).

FIGURE 3.  $(L_\infty, \gamma)$ FIGURE 4.  $L_0$ 

Otherwise, we have  $\deg Q_L \leq \deg Q_{L_0} + 1 = |p|$ . Since  $\det L_\infty \geq 4$  by Lemma 2.2, we have  $|p| = \det L_0 \leq \det L - 4$ . Hence  $\deg Q_L \leq \det L - 4$ .

(3) Both of  $L_0$  and  $L_\infty$  are  $(2, n)$ -torus links.

We assume that  $L_\infty$  is the  $(2, p)$ -torus link, and  $L_0$  is the  $(2, q)$ -torus link. Moreover, we may assume that  $|p| \leq |q|$ . Then

$$\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1 = |q| = \det L - |p|.$$

Since  $L_\infty$  is quasi-alternating,  $p \neq 0$ . If  $|p| \neq 1$ , then we have  $\deg Q_L \leq \det L - 2$ . If  $|p| = 1$ , then  $L_\infty$  is the unknot. Then, as in the proof of Lemma 2.3, take an unknotted arc  $\gamma$  connecting the strands at the resolution of  $L_\infty$ . Let  $K$  be the lift of  $\gamma$  in  $\Sigma(L_\infty) = S^3$ . Then an integral Dehn surgery on  $K$  yields  $\Sigma(L_0) = L(q, 1)$ . We remark that  $|q| = \det L - 1 = m \geq 4$ . By [11, Theorem 1.1] and [19, Theorem 9],  $K$  is the unknot, or a trefoil.

Assume that  $K$  is the unknot. Then, since  $\Sigma(L)$  is obtained by an integral Dehn surgery on  $K$ , it is a lens space  $\pm L(r, 1)$ , with  $r = \det L$ . But this implies that  $L$  is the  $(2, \pm r)$ -torus link by [9], a contradiction.

Finally, assume that  $K$  is a trefoil. By [19, Theorem 9],  $|q| = 5$ . Thus we have  $\det L = 6$ . For a trefoil  $K$  in  $\Sigma(L_\infty)$ , it is well known that there is the unique inverting involution (see [18]). We may assume that  $K$  is right-handed. By taking the quotient of  $(\Sigma(L_\infty), K)$  under the involution, we can recover  $\gamma$  as in Figure 3, with ignoring the framing of  $\gamma$ .

Since  $L_0$  is obtained from  $L_\infty$  by banding along  $\gamma$ , we have Figure 4, where  $k$  denotes the number of half-twists. (If  $k \geq 0$ , then the twists are right-handed. Otherwise, left-handed.)

Then  $L_0$  is the pretzel link of type  $(2, -3, k-2)$ . Since  $L_0$  is 2-bridge,  $|k-2| \leq 1$ . Hence  $k = 1, 2$  or  $3$ . The only possibility for  $L_0$  to be the  $(2, \pm 5)$ -torus knot is  $k = 1$ . Then  $L$  is the pretzel link of type  $(2, 3, 0)$  or  $(2, 3, -2)$ . The former

gives the connected sum of a trefoil and the Hopf link, so  $\deg Q_L = 3$ . Hence  $\deg Q_L = \det L - 3$ . The latter has determinant 4, contradicting  $\det L = 6$ .  $\square$

**Example 2.4.** Let  $K$  be the knot  $10_{140}$  in the knot table. It is hyperbolic and has determinant 9. But the  $Q$ -polynomial is  $2x^8 + 4x^7 - 12x^6 - 22x^5 + 24x^4 + 32x^3 - 24x^2 - 12x + 9$ , so  $K$  is not quasi-alternating by Theorem 1.2. The evaluation (Theorem 1.1) of Qazaqzeh and Chbili [16] cannot detect this fact. We remark that this knot is known to be non-quasi-alternating, because it has thick odd Khovanov homology (see [4, p.2456]).

Also, among 11, 12-crossing non-alternating knots expressed in Dowker-Thistlethwaite notation,  $12_{n0025}, 12_{n0093}, 12_{n0115}, 12_{n0138}, 12_{n0199}, 12_{n0321}, 12_{n0355}, 12_{n0374}, 12_{n0433}, 12_{n0457}, 12_{n0648}$  have determinant 11, but the degree of their  $Q$ -polynomials is 10 (see [3]). Thus these are not quasi-alternating. Again, this fact was confirmed in [10] by using homologically thickness.

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